Supplementary Material to: Realizations of a Special Class of Admittances with Strictly Lower Complexity than Canonical Forms

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***Abstract*—This is supplementary material to "Realizations of a special class of admittances with strictly lower complexity than canonical forms" [1], which presents the detailed proofs of some results. For more background information, refer to [2]–[22] and references therein.**

***Index Terms*—Network synthesis, transformerless synthesis, canonical realization.**

I. Introduction

This report presents the proofs of some results in the paper "Realizations of a special class of admittances with strictly lower complexity than canonical forms" [1]. It is assumed that the numbering of lemmas, theorems, corollaries, and figures in this document agrees with that in the original paper.

II. Proof of Lemma 2

Proof: Sufficiency. It suffices to prove that the three inequalities of Lemma 1 hold. For Case 1, $d_0 = 0$ makes the three inequalities be equivalent to $a_0 d_1 \geq 0$, $a_0 \geq 0$, and $a_1 - d_1 \geq 0$, which are obviously satisfied because of $a_1 - d_1 \geq 0$ and the assumption that all the four coefficients be nonnegative. For Case 2, $a_1 = 0$, $d_1 = 0$ make the three inequalities be equivalent to $a_0 - d_0 \geq 0$, which obviously holds.

Necessity. Suppose that $Y(s)$ is positive-real with at least one of the four coefficients being zero. If the number of zero coefficients is exactly one, then to ensure the three inequalities of Lemma 1 to hold, it is noted that only $d_0 = 0$ is possible. In this case, a necessary and sufficient condition for $Y(s)$ to be positive-real is $a_1 - d_1 \geq 0$. If the number of zero coefficients is exactly two, then to guarantee the positive-realness of $Y(s)$ only the cases when (1) $a_0 = 0$ and $d_0 = 0$; (2) $a_1 = 0$ and $d_1 = 0$; (3) $d_0 = 0$ and $d_1 = 0$ are possible. $Y(s)$ is positive-real if and only if $a_1 - d_1 \geq 0$ when $a_0 = 0$ and $d_0 = 0$. $Y(s)$ is positive-real if and only if $a_0 - d_0 \geq 0$ when $a_1 = 0$ and $d_1 = 0$. $Y(s)$ must always be positive-real when $d_0 = 0$ and $d_1 = 0$, implying that $a_1 - d_1 \geq 0$ must hold. If the number of zero coefficients is exactly three, then to ensure the positive-realness only the cases when (1) $a_0 = 0$, $d_0 = 0$, and $d_1 = 0$; (2) $a_1 = 0$, $d_0 = 0$, and $d_1 = 0$ are possible. It is obvious that the two cases must always be positive-real, which implies $a_1 - d_1 \geq 0$. If the four coefficients are all zero, then $Y(s)$ is always positive-real, and $a_1 - d_1 \geq 0$ always holds. Summarizing all the above discussion, the two cases in the theorem are obtained. ■

III. Proof of Theorem 1

Proof: Based on Lemma 2, it is obvious that there are two possible cases. For Case 1, since $d_0 = 0$ and $a_1 - d_1 \geq 0$, $Y(s)$ can be written as

$$Y(s) = k\frac{a_0 s^2 + a_1 s + 1}{d_1 s^2 + s},$$

and $R_k = a_0(a_0 + d_1^2 - a_1 d_1)$. If $R_k > 0$, then $Y(s)$ is written as

$$\begin{aligned} Y(s) =& k/s + (1/(ka_0/d_1) \\ & + 1/\left(ka_0^3 s/R_k + ka_0^2(a_1 - d_1)/R_k\right))^{-1}, \end{aligned}$$

which is realizable with one inductor, one capacitor, and at most two resistors. If $R_k < 0$, then $Y(s)$ can be written as

$$Y(s) = k/s + ka_0/d_1 + (-a_0 d_1/(kR_k) - a_0 d_1^2 s/(kR_k))^{-1},$$

which is realizable with two inductors and two resistors. For Case 2, since $a_1 = 0$, $d_1 = 0$, and $a_0 - d_0 \geq 0$, $Y(s)$ can be written as

$$Y(s) = k\frac{a_0 s^2 + 1}{s(d_0 s^2 + 1)} = \frac{k}{s} + \frac{k(a_0 - d_0)s}{d_0 s^2 + 1}$$

and $R_k = (a_0 - d_0)^2$. Since it is assumed that $R_k \neq 0$, we conclude that $R_k > 0$ because of the positive-realness of $Y(s)$. $Y(s)$ is realizable with two inductors and one capacitor, which is of strictly lower complexity than the canonical network. ■

IV. Proof of Lemma 3

Proof: Suppose that $Y(s)$ can be realized by the lossless network, then by [23] the even part of $Y^{-1}(s)$ is equal to zero, that is, $Ev\ Y^{-1}(s) = 0$. It therefore follows that

$$\begin{aligned} Ev\ Y^{-1}(s) &= \frac{1}{2}\left(Y^{-1}(s) + Y^{-1}(-s)\right) \\ &= \frac{2s^2((a_0 d_1 - a_1 d_0)s^2 + (d_1 - a_1))}{k(a_0 s^2 + a_1 s + 1)(a_0 s^2 - a_1 s + 1)} = 0. \end{aligned}$$

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Thus, $2s^2((a_0d_1 - a_1d_0)s^2 + (d_1 - a_1)) = 0$ holds for all s. Then we have $a_1 - d_1 = 0$ and $a_0d_1 - a_1d_0 = 0$, which indicates that $Y(s) = k/s$. ■

V. Proof of Lemma 4

Proof: It is obvious that $Z(s) = Y^{-1}(s)$ has a pole at $s = \infty$ and a zero at $s = 0$. By [24, Theorem 2], this lemma can be easily proven. ■

VI. Proof of Lemma 5

Proof: For the network in Fig. 3(b), we see that there must exist poles on the imaginary axis $s = j\omega_0$ with $\omega_0 \neq 0$, which contradicts the fact that all the coefficients be positive. Since any network in the form of Fig. 3(a) is the frequency-inverse dual of a network in Fig. 3(b), then it cannot realize this class of admittances, either. ■

VII. Proof of Lemma 7

Proof: It has been discussed that $Y_1(s)$ can be written as $Y_1(s) = Y^{-1}(s^{-1}) = k'(a_0's^2 + a_1's + 1)/\left(s(d_0's^2 + d_1's + 1)\right)$ with $a_0' = 1/d_0$, $a_1' = d_1/d_0$, $d_0' = 1/a_0$, and $d_1' = a_1/a_0$. Calculating the corresponding R_{k_1}, we obtain $R_{k_1} = \left((a_0 - d_0)^2 - (a_1 - d_1)(a_0d_1 - a_1d_0)\right)/(a_0^2d_0^2) = R_k/(a_0^2d_0^2)$. Thus, this completes the proof. ■

VIII. Proof of Lemma 8

Sufficiency. Since $R_k = 0$, there must be at least one common factor between $(a_0s^2 + a_1s + 1)$ and $(d_0s^2 + d_1s + 1)$. Therefore, $Y(s)$ must be of the form

$$Y(s) = k\frac{(As+1)(Bs+1)}{s(Cs+1)(Bs+1)} = k\frac{As+1}{s(Cs+1)},$$

where A, B, $C > 0$. Comparing the above equation with

$$Y(s) = k\frac{a_0s^2 + a_1s + 1}{s(d_0s^2 + d_1s + 1)}, \tag{VIII.1}$$

where a_0, a_1, d_0, $d_1 \geq 0$ and $k > 0$, we have the following relations: $a_0 = AB$, $a_1 = A+B$, $d_0 = BC$, and $d_1 = B+C$. Then we obtain $a_0/d_0 = A/C$. Since $Y(s)$ is positive-real, it is therefore implied that $a_0 - d_0 \geq 0$, which indicates that $A \geq C$. If $A = C$, then $Y(s)$ reduces to $Y(s) = k/s$, realizable as just an inductor. Otherwise, $A > C$ leads to

$$Y(s) = k\frac{As+1}{s(Cs+1)} = \frac{k}{s} + \frac{1}{\frac{Cs}{k(A-C)} + \frac{1}{k(A-C)}},$$

which is realizable with three elements.

Necessity. By the method of enumeration, one half of network graphs of two-terminal networks with at most three elements are shown in Fig. S1. Lemma 4 implies that the network with the network graph shown in Fig. S1(a) can only realize the network in Fig. 5(a). If we disregard the networks which can always be reduced to those with less elements, then graphs in Fig. S1(b) and Fig. S1(c) are immediately eliminated, because all the elements could only be inductors by Lemma 4. Then based on Lemma 5, we conclude that only the network shown in Fig. 5(b) is possible, which is equivalent to its frequency-inverse dual as shown in Fig. 5(c). In summary, $Y(s)$ can be realized as the admittance of at least one of the networks shown in Fig. 5. By calculating their admittances, we see that $R_k = 0$.

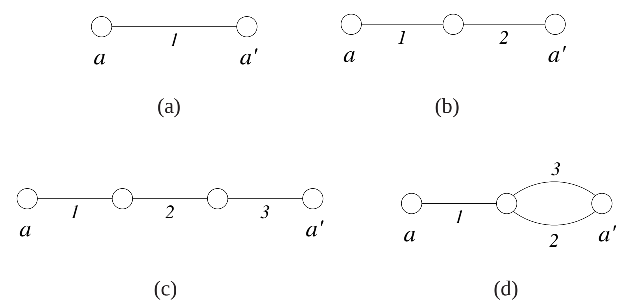

Fig. S1. One half of the network graphs of the two-terminal networks with at most three elements.

IX. Proof of Theorem 2

Proof: Sufficiency. For the network shown in Fig. 7(a), its admittance is calculated as

$$Y(s) = \frac{L_2C_1s^2 + R_1C_1s + 1}{s(L_1L_2C_1s^2 + R_1C_1(L_1 + L_2)s + L_1 + L_2)},$$

which can be expressed in the form of (VIII.1), where $a_0 = L_2C_1 > 0$, $a_1 = R_1C_1 > 0$, $d_0 = L_1L_2C_1/(L_1 + L_2) > 0$, $d_1 = R_1C_1 > 0$, and $k = 1/(L_1 + L_2) > 0$. Then it is calculated that $R_k = C_1^2L_2^4/(L_1 + L_2)^2 \neq 0$. So is the admittance of its frequency-inverse dual shown in Fig. 7(b) by Lemma 7, and hence the equivalent networks in Fig. 7(c) and Fig. 7(d). Thus, the sufficiency part of this theorem is proven.

Necessity. Suppose that $Y(s)$ can be realized with at most four elements whose values are positive and finite. Based on Lemma 8, $R_k \neq 0$ guarantees that we only need to consider the irreducible four-element network. The method of enumeration is used here for the proof. One half of the network graphs of the two-terminal four-element networks are listed in Fig. S2, and other possible graphs are dual with them.

If a four-element network can always be equivalent to one with less elements, then by Lemma 8 we know $R_k = 0$, contradicting with the assumption. It has been stated in Lemma 4 that there must be a path $\mathcal{P}(a, a')$ and a cut-set $\mathcal{C}(a, a')$ consisting of only inductors for the possible realizations. If the networks whose graphs in Fig. S2(a), Fig. S2(b), and Fig. S2(e) satisfy this property, then they must be equivalent to the ones with at most three elements. Therefore, all these network graphs should be eliminated. For Fig. S2(c), Edge 1 and Edge 4 must be inductors by Lemma 4 and Lemma 8. Since $R_k \neq 0$, $Y(s)$ cannot be written as $Y(s) = k/s$, which means that lossless networks do not need to be considered by Lemma 3. Then there must be exactly one resistor either on Edge 2 or on Edge 3. If the other element is the inductor, then we obtain the network shown in Fig. 6, which by Lemma 9 means that $R_k = 0$. Then, it is eliminated. Therefore, only the network in Fig. 7(a) is possible for the graph in Fig. S2(c). Using a similar argument, we have that only the network in Fig. 7(d)

is possible for graph in Fig. S2(d). Using the frequency-inverse dual operation, the networks in Fig. 7(b) and Fig. 7(c) are obtained. It is noted that by Lemma 6 the networks in Fig. 7(a) and Fig. 7(b) are equivalent to those in Fig. 7(c) and Fig. 7(d), respectively. So far, all the possible networks have been discovered. ■

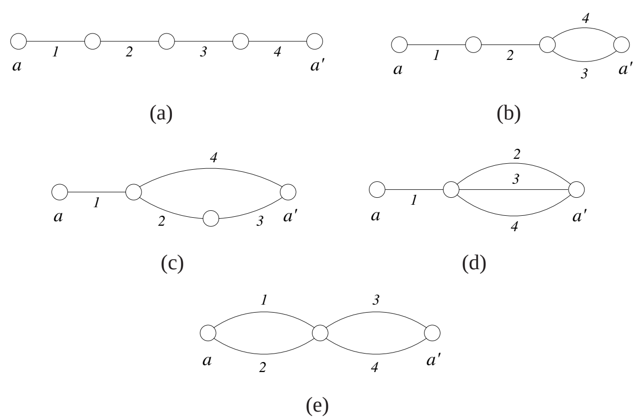

Fig. S2. One half of the network graphs of the two-terminal networks with four elements.

X. Proof of Theorem 3

Theorem 3: Consider an admittance $Y(s)$ in the form of (VIII.1) with a_0, a_1, d_0, d_1, $k > 0$ and $R_k \neq 0$. Then it can be realized by the network shown in Fig. 7(a) with all the elements positive and finite if and only if

$$a_0 - d_0 > 0, \tag{X.1}$$

$$a_1 - d_1 = 0. \tag{X.2}$$

Moreover, if the condition is satisfied, then the values of elements are expressed as

$$R_1 = \frac{a_1(a_0 - d_0)}{k a_0^2}, \tag{X.3}$$

$$L_1 = \frac{d_0}{k a_0}, \tag{X.4}$$

$$L_2 = \frac{a_0 - d_0}{k a_0}, \tag{X.5}$$

$$C_1 = \frac{k a_0^2}{a_0 - d_0}. \tag{X.6}$$

Proof: Necessity. The admittance of the network shown in Fig. 7(a) can be calculated as

$$Y(s) = \frac{L_2 C_1 s^2 + R_1 C_1 s + 1}{s(L_1 L_2 C_1 s^2 + R_1 C_1 (L_1 + L_2) s + L_1 + L_2)}.$$

Suppose that it can realize admittance $Y(s)$ in the form of (VIII.1) with a_0, a_1, d_0, d_1, $k > 0$, then we obtain

$$a_0 = L_2 C_1, \tag{X.7}$$

$$a_1 = R_1 C_1, \tag{X.8}$$

$$d_0 = \frac{L_1 L_2 C_1}{L_1 + L_2}, \tag{X.9}$$

$$d_1 = R_1 C_1, \tag{X.10}$$

$$k = \frac{1}{L_1 + L_2}. \tag{X.11}$$

From (X.8) and (X.10), it is obvious that (X.2) holds. From (X.7) and (X.9), we obtain

$$\frac{a_0}{d_0} = \frac{L_1 + L_2}{L_1},$$

from which and together with (X.11), we obtain that L_1 can be expressed as (X.4). Substituting L_1 into (X.11), L_1 is yielded in the form of (X.5), from which we indicate (X.1). Substituting L_2 into (X.7), we have (X.6). Finally, from (X.8), we obtain (X.3).

Sufficiency. Consider an admittance $Y(s)$ in the form of (VIII.1), where a_0, a_1, d_0, d_1, $k > 0$, $R_k \neq 0$, and (X.1) and (X.2) hold. Calculate R_1, L_1, L_2, and C_1 by (X.3)–(X.6). Since (X.1) holds, it can be verified that all the values of elements are positive and finite. Since (X.2) holds, (X.7)–(X.11) must be satisfied. Therefore, the admittance of the network in Fig. 7(a) equals (VIII.1), which implies the sufficiency. ■

XI. Proof of Theorem 4

Proof: Sufficiency. By Lemma 8, the realization of Case 1 can obviously be satisfied when $R_k = 0$. It is not difficult to see that the condition of Case 2 is equivalent to $a_0 - d_0 > 0$, and either $a_1 - d_1 = 0$ or $a_0 d_1 - a_1 d_0 = 0$. When $a_0 - d_0 > 0$ and $a_1 - d_1 = 0$, $Y(s)$ can be realized as in Fig. 7(a) by Theorem 3, which is a four-element network. Through the use of frequency-inverse dual, the case can be proven when $a_0 - d_0 > 0$ and $a_0 d_1 - a_1 d_0 = 0$.

Necessity. Suppose that admittance $Y(s)$ in the form of (VIII.1) with a_0, a_1, d_0, d_1, $k > 0$ can be realized with at most four elements, whose values are positive and finite. We then divide it into two cases: $R_k = 0$ and $R_k \neq 0$. It is known from Lemma 8 that the case when $R_k = 0$ always holds. When $R_k \neq 0$, it is seen from Theorem 2 that $Y(s)$ is the admittance of the network shown in Fig. 7. Together with the realizability conditions derived above, we obtain $a_0 - d_0 > 0$, and either $a_1 - d_1 = 0$ or $a_0 d_1 - a_1 d_0 = 0$, implying Case 2. ■

XII. Proof of Theorem 5

Proof: Necessity. Since $R_k \neq 0$, then $Y(s)$ must be an RL (SD) admittance of McMillan degree three. By [22, Theorem 4], the coefficients must satisfy $R_k < 0$.

Sufficiency. Since $R_k := (a_0 - d_0)^2 - (a_0 d_1 - a_1 d_0)(a_1 - d_1) < 0$, then we have $a_0 d_1 - a_1 d_0 > 0$. In addition, we know $d_0, d_1 > 0$, then it is known from [22, Theorem 4] that $Y(s)$ is an RL (SD) admittance of McMillan degree three. From [22], [25], $Y(s)$ can be of the form

$$Y(s) = k \frac{(As + 1)(Cs + 1)}{s(Bs + 1)(Ds + 1)},$$

where $A > B > C > D > 0$ and $k > 0$. Therefore, it is the admittance of a network with two resistors and an arbitrary number of inductors. Furthermore, $Y(s)$ can be written as

$$Y(s) = \frac{k}{s} + \frac{k(A - B)(B - C)}{(B - D)(Bs + 1)} + \frac{k(A - D)(C - D)}{(B - D)(Ds + 1)},$$

which is obviously the admittance of the network in Fig. 8 with the values satisfying $L_1 = 1/k$, $L_2 = B(B - D)/(k(A - B)(B - C))$, $L_3 = D(B - D)/(k(A - D)(C - D))$,

$R_1 = (B-D)/(k(A-B)(B-C))$, $R_2 = (B-D)/(k(A-D)(C-D))$, where $A, C = \left(a_1 \pm \sqrt{a_1^2 - 4a_0}\right)/2$ and $B, D = \left(d_1 \pm \sqrt{d_1^2 - 4d_0}\right)/2$. Since $a_0 = AC$, $a_1 = A+C$, $d_0 = BD$, and $d_1 = B + D$, then by solving them we obtain the expressions of A, B, C, and D as stated in this theorem. ■

XIII. Proof of Lemma 10

Proof: For Fig. 9(a), the admittance can be calculated as $Y(s) = (C_1L_2L_3s^3 + R_1C_1L_3s^2 + (L_2 + L_3)s + R_1)/(s(C_1L_1L_2L_3s^3 + R_1C_1L_1L_3s^2 + (L_1L_2 + L_2L_3 + L_1L_3)s + R_1(L_1 + L_3)))$. Since $R_k \neq 0$, then $Y(s)$ can be realized as in Fig. 9(a), if and only if there exists $T > 0$ such that the following equations hold

$$a_0T = \frac{C_1L_2L_3}{R_1},\ a_0 + a_1T = C_1L_3,\ a_1 + T = \frac{L_2+L_3}{R_1}, \tag{XIII.1}$$

$$d_0T = \frac{C_1L_1L_2L_3}{R_1(L_1+L_3)}, \quad d_0 + d_1T = \frac{C_1L_1L_3}{L_1+L_3}, \tag{XIII.2}$$

$$d_1 + T = \frac{L_1L_2 + L_2L_3 + L_1L_3}{R_1(L_1+L_3)}, \quad k = \frac{1}{L_1+L_3}. \tag{XIII.3}$$

Therefore, it suffices to show that $T > 0$ does not exist. After a series of calculations, it is verified that (XIII.1)–(XIII.3) are equivalent to

$$\begin{aligned} R_1 &= \frac{(a_0+a_1T)(a_0-d_0)}{ka_0a_1(T^2+a_1T+a_0)} \\ &= \frac{(a_0-d_0)(a_1d_0T^2 + (a_0^2 + a_1^2d_0)T + a_0a_1d_0)}{ka_0^2a_1(d_1+T)(T^2+a_1T+a_0)}, \end{aligned} \tag{XIII.4}$$

$$L_1 = \frac{d_0}{ka_0},\ L_2 = \frac{(a_0-d_0)T}{ka_1(T^2+a_1T+a_0)},\ L_3 = \frac{a_0-d_0}{ka_0}, \tag{XIII.5}$$

$$C_1 = \frac{ka_0(a_0+a_1T)}{a_0-d_0} = \frac{ka_0^2(d_0+d_1T)}{d_0(a_0-d_0)}. \tag{XIII.6}$$

From (XIII.6), we obtain $a_0 - d_0 > 0$ and $a_1d_0 - a_0d_1 = 0$. The second equality of (XIII.4) is equivalent to $a_1(a_0-d_0)T^2 = 0$, which obviously cannot hold for any $T > 0$. Therefore, there does not exist any $T > 0$ such that (XIII.1)–(XIII.3) hold simultaneously.

Using a similar argument, the conclusion of the network in Fig. 9(c) can also be proven. Furthermore, since the networks in Fig. 9(b) and Fig. 9(d) are the frequency-inverse duals of those in Fig. 9(a) and Fig. 9(c), respectively, they cannot be realized, either. ■

XIV. Proof of Lemma 12

Lemma 12: Consider a positive-real function $Y(s)$. Then it can be realized as the admittance of the network shown in Fig. 12 with the values of the elements being positive and finite, if and only if $Y(s)$ can be expressed as

$$Y(s) = \frac{\alpha_3s^3 + \alpha_2s^2 + \alpha_1s + 1}{\beta_4s^4 + \beta_3s^3 + \beta_2s^2 + \beta_1s}, \tag{XIV.1}$$

where α_1, α_2, α_2, β_1, β_2, β_3, $\beta_4 > 0$, W_1, W_2, W_3, $W - 2\alpha_2W_3 > 0$, and the following two equations hold

$$W^2 - 4W_1W_2W_3 = 0, \tag{XIV.2}$$

$$\beta_4 + \alpha_1\beta_3 + \alpha_3\beta_1 - \alpha_2\beta_2 = 0. \tag{XIV.3}$$

Moreover, if the conditions hold, then the values of the network can be expressed as follows

$$R_1 = \frac{(\alpha_1\alpha_2 - \alpha_3)\beta_1^2}{\alpha_1^2(\alpha_2\beta_1 - \beta_3)}, \tag{XIV.4}$$

$$L_1 = \frac{(\alpha_1\alpha_2\beta_1 - \alpha_3\beta_1 - \alpha_1\beta_3)\beta_1}{\alpha_1(\alpha_2\beta_1 - \beta_3)}, \tag{XIV.5}$$

$$L_2 = \frac{\alpha_3\beta_1^2}{\alpha_1(\alpha_2\beta_1 - \beta_3)}, \tag{XIV.6}$$

$$L_3 = \frac{\beta_1\beta_3}{\alpha_2\beta_1 - \beta_3}, \tag{XIV.7}$$

$$C_1 = \frac{\alpha_2\beta_1 - \beta_3}{\beta_1^2}. \tag{XIV.8}$$

Proof: Necessity. Calculate the admittance of the network in Fig. 12, then we obtain $Y(s) = (C_1L_2(L_1+L_3)s^3 + R_1C_1(L_1+L_2+L_3)s^2 + (L_1+L_3)s + R_1)/(s(C_1L_1L_2L_3s^3 + R_1C_1L_3(L_1+L_2)s^2 + (L_1L_2 + L_2L_3 + L_1L_3)s + R_1(L_1+L_2)))$. Therefore, $Y(s)$ can be expressed as (XIV.1) with the coefficients satisfying

$$\alpha_3 = \frac{C_1L_2(L_1+L_3)}{R_1}, \quad \alpha_2 = C_1(L_1+L_2+L_3), \tag{XIV.9}$$

$$\alpha_1 = \frac{L_1+L_3}{R_1},\ \beta_4 = \frac{C_1L_1L_2L_3}{R_1},\ \beta_3 = C_1L_3(L_1+L_2), \tag{XIV.10}$$

$$\beta_2 = \frac{L_1L_2 + L_2L_3 + L_1L_3}{R_1}, \quad \beta_1 = L_1 + L_2. \tag{XIV.11}$$

It is obvious that α_1, α_2, α_2, β_1, β_2, β_3, $\beta_4 > 0$. After a series of calculations, the above equations are equivalent to (XIV.4)–(XIV.8) and the following two equations

$$\begin{aligned} &\alpha_1\alpha_2^2\beta_1\beta_4 + \alpha_3\beta_3\beta_4 + \alpha_3^2\beta_1\beta_3 + \alpha_1\alpha_3\beta_3^2 \\ &\quad = \alpha_2\alpha_3\beta_1\beta_4 + \alpha_1\alpha_2\beta_3\beta_4 + \alpha_1\alpha_2\alpha_3\beta_1\beta_3, \end{aligned} \tag{XIV.12}$$

$$\begin{aligned} &\alpha_1\alpha_2^2\beta_1\beta_2 + \alpha_3\beta_2\beta_3 + \alpha_3^2\beta_1^2 + \alpha_1\alpha_3\beta_1\beta_3 + \alpha_1^2\beta_3^2 \\ &\quad = \alpha_2\alpha_3\beta_1\beta_2 + \alpha_1\alpha_2\beta_2\beta_3 + \alpha_1\alpha_2\alpha_3\beta_1^2 + \alpha_1^2\alpha_2\beta_1\beta_3. \end{aligned} \tag{XIV.13}$$

Furthermore, (XIV.13) is equivalent to $\alpha_1\alpha_2\alpha_3\beta_1\beta_3 - \alpha_3^2\beta_1\beta_3 - \alpha_1\alpha_3\beta_3^2 = (\alpha_1\alpha_2^2\beta_1 + \alpha_3\beta_3 - \alpha_2\alpha_3\beta_1 - \alpha_1\alpha_2\beta_3)(\alpha_2\beta_2 - \alpha_1\beta_3 - \alpha_3\beta_1)$. Since all the values of the elements are positive and finite, then we can calculate that $W_1 = \alpha_1\alpha_2 - \alpha_3 = C_1(L_1+L_3)^2/R1 > 0$, $W_2 = \alpha_2\beta_1 - \beta_3 = C_1(L_1+L_2)^2 > 0$, $W_3 = L_1^2/R_1 > 0$, and

$W - 2\alpha_2 W_3 = 2C_1L_1L_2L_3/R_1 > 0$. Since W_1, $W_2 > 0$, then

$$\begin{aligned}\alpha_1\alpha_2^2\beta_1 + \alpha_3\beta_3 - \alpha_2\alpha_3\beta_1 - \alpha_1\alpha_2\beta_3 \\ = (\alpha_1\alpha_2 - \alpha_3)(\alpha_2\beta_1 - \beta_3) = W_1W_2 > 0.\end{aligned} \tag{XIV.14}$$

Therefore, it follows from (XIV.12) that

$$\begin{aligned}\beta_4 =& \frac{\alpha_1\alpha_2\alpha_3\beta_1\beta_3 - \alpha_3^2\beta_1\beta_3 - \alpha_1\alpha_3\beta_3^2}{\alpha_1\alpha_2^2\beta_1 + \alpha_3\beta_3 - \alpha_2\alpha_3\beta_1 - \alpha_1\alpha_2\beta_3} \\ =& \frac{(\alpha_1\alpha_2^2\beta_1 + \alpha_3\beta_3 - \alpha_2\alpha_3\beta_1 - \alpha_1\alpha_2\beta_3)}{\alpha_1\alpha_2^2\beta_1 + \alpha_3\beta_3 - \alpha_2\alpha_3\beta_1 - \alpha_1\alpha_2\beta_3} \\ & \times (\alpha_2\beta_2 - \alpha_1\beta_3 - \alpha_3\beta_1) \\ =& \alpha_2\beta_2 - \alpha_1\beta_3 - \alpha_3\beta_1,\end{aligned} \tag{XIV.15}$$

which implies (XIV.3). Substituting the above equation into W, we obtain $W = 2(\alpha_1\alpha_2\beta_1 - \alpha_1\beta_3 - \alpha_3\beta_1)$, implying

$$\begin{aligned}&4(\alpha_1\alpha_3\beta_1\beta_3 - \alpha_1^2\alpha_2\beta_1\beta_3 - \alpha_1\alpha_2\alpha_3\beta_1^2 + \alpha_1^2\beta_3^2 + \alpha_3^2\beta_1^2 \\ &\quad + \alpha_1\alpha_2^2\beta_1\beta_2 - \alpha_2\alpha_3\beta_1\beta_2 - \alpha_1\alpha_2\beta_2\beta_3 + \alpha_3\beta_2\beta_3) \\ =&4(\alpha_1\alpha_2\beta_1 - \alpha_3\beta_1 - \alpha_1\beta_3)^2 \\ &- 4(\alpha_1\alpha_2 - \alpha_3)(\alpha_2\beta_1 - \beta_3)(\alpha_1\beta_1 - \beta_2) \\ =&W^2 - 4W_1W_2W_3.\end{aligned} \tag{XIV.16}$$

Now, (XIV.13) and (XIV.16) yield (XIV.2).

Sufficiency. Let the values of R_1, L_1, L_2, L_3, and C_1 satisfy (XIV.4)–(XIV.8). W_1, $W_2 > 0$ indicates that R_1, L_2, L_3, $C_1 > 0$. Then substituting β_4 obtained from (XIV.3) into W, we obtain $W - 2\alpha_2W_3 = 2(\alpha_2\beta_2 - \alpha_1\beta_3 - \alpha_3\beta_1)$. Since $W - 2\alpha_2W_3 > 0$, it follows that $L_1 > 0$. Substituting β_4 obtained from (XIV.3) into (XIV.2), we obtain Equation (XIV.13) immediately. Together with (XIV.14) and (XIV.15), we obtain Equation (XIV.12). It is known in the necessity part that (XIV.9)–(XIV.11) must hold. Now, the sufficiency is proven. ■

XV. Proof of Lemma 13

Lemma 13: Consider any positive-real function $Y(s)$ in the form of (VIII.1) with a_0, a_1, d_0, d_1, $k > 0$, and $R_k \neq 0$. Then it can be realized as the admittance of the network as shown in Fig. 12 with the values of the elements being positive and finite if and only if

$$(a_0d_1 - a_1d_0)(a_1 - d_1) - d_0^2 = 0. \tag{XV.1}$$

Moreover, the values of the elements can be expressed as

$$R_1 = \frac{a_1(T^2 + a_1T + a_0)}{k(a_1 + T)^2\left((a_1 - d_1)T + (a_0 - d_0)\right)}, \tag{XV.2}$$

$$L_1 = \frac{(a_1 - d_1)T^2 + (a_1^2 - a_1d_1 - d_0)T + a_1(a_0 - d_0)}{k(a_1 + T)\left((a_1 - d_1)T + (a_0 - d_0)\right)}, \tag{XV.3}$$

$$L_2 = \frac{a_0T}{k(a_1 + T)\left((a_1 - d_1)T + (a_0 - d_0)\right)}, \tag{XV.4}$$

$$L_3 = \frac{d_1T + d_0}{k\left((a_1 - d_1)T + (a_0 - d_0)\right)}, \tag{XV.5}$$

$$C_1 = k\left((a_1 - d_1)T + (a_0 - d_0)\right), \tag{XV.6}$$

where

$$T = \sqrt{\frac{a_0d_1 - a_1d_0}{a_1 - d_1}}. \tag{XV.7}$$

Proof: Necessity. Suppose that $Y(s)$ in the form of (VIII.1) with all the coefficients positive can be realized as the admittance of the network shown in Fig. 12. It then follows from Lemma 12 that $Y(s)$ can be expressed as (XIV.1) with all the coefficients positive and satisfying the condition of Lemma 12. Since $R_k \neq 0$, the only way to express (VIII.1) as (XIV.1) where α_1, α_2, α_2, β_1, β_2, β_3, $\beta_4 > 0$ is to multiply the numerator and denominator with $(Ts + 1)$ where $T > 0$. Consequently, it follows that

$$\begin{aligned}\alpha_3 = a_0T,\ \alpha_2 = a_0 + a_1T,\ \alpha_1 = a_1 + T, \\ \beta_4 = \frac{d_0T}{k},\ \beta_3 = \frac{d_0 + d_1T}{k},\ \beta_2 = \frac{d_1 + T}{k},\ \beta_1 = \frac{1}{k}.\end{aligned} \tag{XV.8}$$

It is obvious that the condition that α_1, α_2, α_2, β_1, β_2, β_3, $\beta_4 > 0$ always holds. Furthermore, other conditions can be presented as follows

$$W_1 = \alpha_1\alpha_2 - \alpha_3 = a_1T^2 + a_1^2T + a_0a_1 > 0, \tag{XV.9}$$

$$W_2 = \alpha_2\beta_1 - \beta_3 = \frac{(a_1 - d_1)T + (a_0 - d_0)}{k} > 0, \tag{XV.10}$$

$$W_3 = \alpha_1\beta_1 - \beta_2 = \frac{a_1 - d_1}{k} > 0, \tag{XV.11}$$

$$W - 2\alpha_2W_3 = \frac{(a_1 - d_1)T^2 + (a_0d_1 - a_1d_0)}{k} > 0, \tag{XV.12}$$

$$W^2 - 4W_1W_2W_3 = \frac{\left((a_1 - d_1)T^2 - (a_0d_1 - a_1d_0)\right)^2}{k^2} = 0, \tag{XV.13}$$

$$\begin{aligned}\beta_4 + \alpha_1\beta_3 + \alpha_3\beta_1 - \alpha_2\beta_2 \\ = \frac{-(a_1 - d_1)T^2 + 2d_0T - (a_0d_1 - a_1d_0)}{k} = 0.\end{aligned} \tag{XV.14}$$

Then (XV.11) leads to $a_1 - d_1 > 0$. Then T can be solved from (XV.13) as (XV.7). The constraint that $T > 0$ yields $a_0d_1 - a_1d_0 > 0$. Substituting the solved T into (XV.14), we obtain (XV.1).

Sufficiency. Suppose that $(a_0d_1 - a_1d_0)(a_1 - d_1) - d_0^2 = 0$ holds, then $d_0 > 0$ and the positive-realness of $Y(s)$ leads to $a_0d_1 - a_1d_0 > 0$ and $a_1 - d_1 > 0$. Then we can let

$$T = \sqrt{\frac{a_0d_1 - a_1d_0}{a_1 - d_1}} > 0.$$

Furthermore, multiplying the numerator and denominator of $Y(s)$ with the factor $(Ts + 1)$, then we can express $Y(s)$ in the form of (XIV.1) with the coefficients satisfying (XV.8). It is obvious that α_1, α_2, α_2, β_1, β_2, β_3, $\beta_4 > 0$ and (XV.9)–(XV.13) hold. Since $(a_0d_1 - a_1d_0)(a_1 - d_1) - d_0^2 = 0$, then (XV.14) is also satisfied. It is concluded that the condition of Lemma 12 holds, therefore $Y(s)$ can be realized as the admittance of the network as shown in Fig. 12 with the elements positive and finite. The expressions for the values of the elements are presented in (XV.2)–(XV.6) derived from (XIV.4)–(XIV.8) with the relation (XV.8). ■

XVI. Proof of Lemma 14

Proof: Calculating the admittance of the network shown in Fig. 13(a) yields $Y_d(s) = (C_1L_2(L_1 + L_3)s^3 + R_1C_1(L_1+L_3)s^2+(L_1+L_2+L_3)s+R_1)/(s(C_1L_1L_2L_3s^3+R_1C_1(L_1L_2 + L_2L_3 + L_1L_3)s^2 + L_3(L_1 + L_2)s + R_1(L_1 + L_2)))$, which is obviously in the form of (XIV.1) with $\alpha_3 = C_1L_2(L_1 + L3)/R_1$, $\alpha_2 = C_1(L_1 + L_3)$, $\alpha_1 = (L_1 + L_2 + L_3)/R_1$, $\beta_4 = C_1L_1L_2L_3/R_1$, $\beta_3 = C_1(L_1L_2 + L_2L_3 + L_1L_3)$, $\beta_2 = L_3(L_1 + L_2)/R_1$, and $\beta_1 = L_1 + L_2$. It can be calculated that the coefficients must satisfy $W - 2\alpha_2W_3 = -2C_1L_2(L_1 + L_2)(L_1 + L_3)/R_1 < 0$. Similarly, for network in Fig. 13(b), its admittance can also be in the form of (XIV.1) where α_1, α_2, α_2, β_1, β_2, β_3, $\beta_4 > 0$ and satisfy $W - 2\alpha_2W_3 < 0$. Assume that $Y(s)$ in the form of (VIII.1) with a_0, a_1, d_0, d_1, $k > 0$ and $R_k \neq 0$ can be realized by a network as in Fig. 13(a) or Fig. 13(b). Consequently, $Y(s)$ must be able to be expressed in the form of (XIV.1) with α_1, α_2, α_2, β_1, β_2, β_3, $\beta_4 > 0$, implying that the coefficients satisfy (XV.8). In the proof of Lemma 13, it is seen that

$$W - 2\alpha_2W_3 = \frac{(a_1 - d_1)T^2 + (a_0d_1 - a_1d_0)}{k} \geq 0,$$

which contradicts with the hypothesis. Thus this lemma is proven. ■

XVII. Conclusion

In this report, the proofs of some results in the original paper [1] have been presented.

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